ELSEVIER
Journal of Geometry and Physics 31 (1999) 35–50
JOURNAL OF GEOMETRY AND PHYSICS

On quantum mechanics of n-particle systems on 2-manifolds – a case study in topology

H.-D. Doebner *, W. Groth, J.D. Hennig
Technical University of Clausthal, Arnold Sommerfeld Institut für Mathematische Physik, Leibnizstrasse 10, 33678 Clausthal-Zellerfeld, Germany

Received 14 May 1998; received in revised form 24 November 1998

Abstract

A system of n particles localized on a smooth manifold P has a topologically nontrivial configuration space M if one assumes that M is built from P via an n-fold product, and that the particles cannot be located at the same point in P at the same time. Because of this property of M, which holds even if P is topologically trivial, the quantization of the system is not unique: there are unitary inequivalent descriptions of its kinematics and dynamics. If the particles are assumed to be identical, further topological effects appear. We study these situations in a unified and strictly geometrical approach and use as an adequate quantization on manifolds M the Borel quantization which is based on Hilbert spaces of square integrable sections of Hermitian line bundles with flat connections. The manifolds M built from $P = \mathbb{R}^2$ or compact 2-manifolds P are discussed in detail for distinguishable and identical particles; the unitarily inequivalent quantizations are classified; for $P = \mathbb{R}^2$ we calculate the flat connections, the kinematics and the Schrödinger equations for the different quantizations. In Appendix A the situation for $P = \mathbb{R}^m$, $m \geq 3$, is given. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Quantum mechanics
1991 MSC: 53C07, 53C80, 81S99, 81V99
Keywords: Quantization; Two-dimensional manifolds; Braids

1. Introduction

Quantum mechanics is a global theory.

Consider a nonrelativistic, classical, finite dimensional system and its (smooth) connected configuration manifold M. The system, containing n distinguishable or identical particles,

* Corresponding author. E-mail: asi@pt.tu-clausthal.de

PII: S0393-0440(98)00070-9

is quantized via a quantization map $\mathcal{Q}$ which maps classical observables of the system into the set $SA(\mathcal{H})$ of selfadjoint operators $\mathbb{A}$ in some Hilbert space $\mathcal{H}$.

The *topology* of M enters $\mathcal{Q}$ because we want to map observables (in particular generalized momentum observables, see Section 2) into the set of *differential* operators. Essentially, our construction splits into the following steps:

(i) Restricting to systems without internal degrees of freedom, we choose a *smooth* measure μ on M and realize the space $\mathcal{H}$ of pure states as $L^2(M \times \mathbb{C}, \mathrm{d}\mu)$, i.e. by μ-square integrable $\mathbb{C}$-valued functions on M (sections of the Cartesian product $M \times \mathbb{C}$).

(ii) For these sections of $M \times \mathbb{C}$, a priori, *no preferred differentiable structure* (i.e. no C^∞ atlas) on $M \times \mathbb{C}$ is distinguished. To quantize observables as *differential* operators on a dense domain $\mathcal{D}$ in $\mathcal{H}$ (as in usual quantum mechanics on $\mathbb{R}^n$) one needs a suitable $\mathcal{D}$, such that differentiation of complex 'functions' on M makes sense (for another argument, based on position observables, see [27]). Hence one has to select a differentiable structure on the set $M \times \mathbb{C}$. A natural selection is to choose a complex smooth *line bundle* $l(M)$ over M (as sets: $M \times \mathbb{C} \equiv$ total space of $l(M)$) with Hermitean product (ψ, ψ') for sections ψ, ψ' of $l(M)$, and to view $\mathcal{H} = L^2(M, \mathrm{d}\mu)$ as the μ-completion of the space of square integrable sections in $l(M) : L^2(l(M), \mathrm{d}\mu)$ (of course, in the 'measurable' category, $l(M)$ is bundle isomorphic to $M \times \mathbb{C}$). The construction of differential operators then requires the specification of a connection ∇ in $l(M)$. On each $l(M)$ connections exist. In particular, for our quantization, *flat* connections are of interest (see Section 2.1). The classes of equivalent pairs $(l(M), \nabla)$ with flat ∇ are in a $1:1$ correspondence to $\pi_1^*(M)$, the characters of the fundamental group of M, i.e. the homomorphisms $\pi_1(M) \to U(1)$. Hence (selfadjoint) quantum observables modelled via differential operators in $L^2(M, \mathrm{d}\mu)$ may depend, together with the spectrum and eigenfunctions, on the topology, i.e. on global properties of M. If the system has internal degrees of freedom, Hermitean higher dimensional *vector bundles* appear (see e.g. [10]).

We discuss this situation in the framework of Borel quantization, sketched in Section 2. We choose the case of a system of n distinguishable or n identical particles, localized on an m-dimensional manifold P, denoted as '*physical space*' or '1-*particle-space*' of the n-particle system. The configuration space M of the system is built through P. We describe the topological situation for general physical spaces P (Section 3) and discuss in detail the Borel quantization on $\mathbb{R}^2$ and on orientable compact 2-manifolds with genus g (Section 4). The quantization map $\mathcal{Q}$ for generalized position and momentum observables is constructed in Section 4.2 for $P = \mathbb{R}^2$, and we determine in Section 4.3 the evolution equation (Schrödinger equation) of the system via Borel quantization.

In our case study a unified and strictly geometrical approach based on Hermitian line bundles is presented. Because of the transparent structure of this formalism new insights are possible and a suggestive view on older results, e.g. on various types of ('exotic') statistics and also on Schrödinger equations for identical particles ('anyons') or distinguishable particles in $P = \mathbb{R}^2$. Anyons could have physical relevance if it is justified to restrict a system of n identical particles in $\mathbb{R}^3$ to 'two dimensions' in a kind of approximation.

Examples are high temperature superconductivity [20] and the fractional quantum Hall effect [2].

Pecularities of configuration spaces for n identical particles in $P = \mathbb{R}^2$ were mentioned already by Fadell and Neuwirth [12] from a mathematical point of view and by Leinaas and Myrheim [21] from a physical one. They were introduced independently into quantum mechanics by Goldin et al. [15], and further related to experimental situations by Wilczek [28,29]; see also [11]. For remarks on the history of anyons we refer to [4,14,22]. The notion of anyons was coined in [29].

2. Borel quantization

2.1. Kinematical part

The quantization of our system starts with a set $\mathcal{O}$ of classical observables, i.e. the subset of 'generalized position observables' which build the real linear space $\mathrm{Fun}(M)$ of smooth functions on M and the subset of 'generalized momentum observables' realized through the set $\mathrm{Vec}(M)$ of smooth vector fields on M. Both $\mathrm{Fun}(M)$ and $\mathrm{Vec}(M)$ are Lie algebras, they couple semidirectly, with ideal $\mathrm{Fun}(M)$, and yield the general symmetry algebra of M, the *kinematical algebra*

$$S(M) = \mathrm{Fun}(M) \oplus_s \mathrm{Vec}(M).$$

For technical reasons we restrict $\mathrm{Vec}(M)$ to complete vector fields, $\mathrm{Vec}_c(M)$. They carry a partial Lie algebra structure since complete vector fields need not yield complete commutators. To construct the quantization map $\mathcal{Q} = (\mathbb{Q}, \mathbb{P}) : S(M) \longrightarrow SA(\mathcal{H})$ with

$$\mathbb{Q} : f \in \mathrm{Fun}(M) \mapsto \mathbb{Q}(f) \in SA(\mathcal{H}),$$
$$\mathbb{P} : X \in \mathrm{Vec}_c(M) \mapsto \mathbb{P}(X) \in SA(\mathcal{H}),$$

we assume (see details in [1,24]):
- $\mathcal{Q}$ is an *isomorphism* into $SA(\mathcal{H})$ with respect to the Lie brackets on $S(M)$ and on $SA(\mathcal{H})$ (the algebraic structure of $S(M)$ should survive $\mathcal{Q}$).
- $\mathcal{H}$ is realized as $L^2(l(M), \mathrm{d}\mu)$ (to have the option to map X to a *differential* operator $\mathbb{P}(X)$ via the choice of a connection ∇ in $l(M)$, as explained in Section 1).
- $\mathbb{P} : X \mapsto \mathbb{P}(X)$ is a *local* map (representing a physical assumption of causality).
- $\mathbb{Q}(f)$ is the multiplication operator (properties of localized position measurements).

One can show that because of the isomorphism property of $\mathcal{Q}$ the connection has to be *flat*, which yields a topological restriction for the possibilities to construct $l(M)$. Locality then implies $\mathbb{P}(X)$ to be a first order differential operator. The representations of $S(M)$ eventually lead to a classification theorem [1]:

Irreducible representations of $S(M)$ in $L^2(l(M), \mathrm{d}\mu)$ are classified by $\pi_1^*(M) \times \mathbb{R}$, where $\pi_1^*(M)$ classifies the line bundles $l(M)$ over M with flat connection [18], and the elements c of $\mathbb{R}$ yield an additional quantum number, not connected to topology, which gives a path to nonlinear quantum mechanics [9]. In this sense $\pi_1^*(M)$ is the gate for the topology of M to enter $\mathcal{Q}$. $\mathbb{Q}$ and $\mathbb{P}$ are given (up to unitary equivalence) on a common dense domain in $L^2(l(M),\, \mathrm{d}\mu)$ as

$$\mathbb{Q}(f) = f, \tag{1}$$

$$\mathbb{P}(X) = \frac{\hbar}{\mathrm{i}}\nabla_X + \left(c + \frac{\hbar}{2\mathrm{i}}\right)\mathrm{div}_\mu\, X, \tag{2}$$

with $c \in \hbar\mathbb{R}$ and ∇ as a flat connection (with local connection 1-form ω on M), which can be viewed as a potential

$$\nabla_X = X + \frac{\mathrm{i}}{\hbar}\omega(X). \tag{3}$$

If $l(M)$ is trivializable, ω on M can be chosen globally, and a necessary and sufficient condition for ω_1 and ω_2 to belong to equivalent ∇_1 and ∇_2 is that $\omega_1 - \omega_2$ is logarithmic exact [18].

This characterizes the quantization of the kinematical algebra of a system on M and gives the quantum Borel *kinematics*.

2.2. *Dynamical part*

We introduce now a time dependence to our system characterized kinematically through an irreducible representation of $S(M)$. There are different [7,8] and physically well-motivated methods which give the same type of constraints for evolution equations for pure states $\psi_t \in L^2(l(M),\, \mathrm{d}\mu)$. Here we use a quantum analogue of the classical relation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(q(t)) = (\mathrm{d}f)(\dot{q}(t)) = (\dot{q},\ \mathrm{grad}_g\, f|_{q(t)}) = p(t)(\mathrm{grad}_g\, f|_{q(t)}) \tag{4}$$

between the time derivative of a function f on M along a path $q(t)$ in M and the momentum p of the system given on M, where M is equipped with a (pseudo-) Riemannian metric g (the masses of the particles are absorbed in g). The analogue of Eq. (4) in $\mathcal{H}$ (with inner product $\langle\ ,\ \rangle$) is written as a relation between expectation values of quantized position observables $\mathbb{Q}(f)$ and momentum observables $\mathbb{P}(X)$, $X = \mathrm{grad}_g\, f$, and reads for pure states (for a detailed version, see [8]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_t, \mathbb{Q}(f)\psi_t\rangle = \langle\psi_t, \mathbb{P}(\mathrm{grad}_g f)\psi_t\rangle \quad \text{for all } f \in \mathrm{Fun}(M). \tag{5}$$

With

$$\rho_t(x) = (\psi_t(x),\, \psi_t(x)), \tag{6}$$

$$j_t^\nabla(x) = \hbar\,\mathrm{Im}(\psi_t(x),\, (\mathrm{grad}_g^\nabla \psi_t)(x)), \tag{7}$$

where grad_g^∇ is the lift of grad_g to $l(M)$ with respect to ∇, i.e. locally

$$(\mathrm{grad}_g^\nabla \psi_t)^j = g^{jk}\left(\frac{\partial}{\partial q^k} + \frac{\mathrm{i}}{\hbar}\omega_k\right)\psi_t, \quad \omega =: \omega_k\, \mathrm{d}q^k,$$

Eqs. (5)–(7) lead to a Fokker–Planck type equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t + \mathrm{div}_g\, j_t^{\nabla} = c\Delta_g\rho_t,$$

which is a condition for the evolution of ψ_t. One easily checks, that this condition implies the nonlinear evolution equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t = \left(-\frac{\hbar^2}{2}\Delta_g^{\nabla} + V\right)\psi_t + \mathrm{i}\frac{\hbar c}{2}\frac{\Delta_g\rho_t}{\rho_t}\psi_t + R[\bar{\psi}_t, \psi_t]\psi_t, \tag{8}$$

with V a real scalar potential and $R[\bar{\psi}, \psi]$ an arbitrary real function of $\bar{\psi}$, ψ and its derivatives, containing possibly also vector or tensor potentials. Locally,

$$\Delta_g^{\nabla} = \frac{1}{\sqrt{|\det g|}}\left(\frac{\partial}{\partial q^j} + \frac{\mathrm{i}}{\hbar}\omega_j\right)\sqrt{|\det g|}g^{jk}\left(\frac{\partial}{\partial q^k} + \frac{\mathrm{i}}{\hbar}\omega_k\right) \tag{9}$$

holds. Note that Δ_g^{∇} is the lift to $l(M)$ of the Laplace–Beltrami operator Δ_g on (M, g).

If one is interested only in the free linear part with potential V, i.e. $c = 0$, $R = 0$, the result is the usual Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t = \left(-\frac{\hbar^2}{2}\Delta_g^{\nabla} + V\right)\psi_t. \tag{10}$$

We use the evolution equation (10) in the sequel.

3. Topological properties of configuration manifolds M for n particles on a physical space P

Our system consists of n distinguishable or n identical particles, each of them being localized on the same physical space P ($m := \dim P \geq 2$). The topological and group theoretical relations between its configuration manifolds M and the corresponding Hermitian line bundle (h.l.b.) on the one side, and the typical (anti-) symmetrization processes on the other, are most transparently described in the language of principal fibre bundles (p.f.b.s):

For the construction of M we accept the following view:

(i) M is built from P via an n-fold product.

(ii) Different (point-like) classical particles cannot be located at the same point in P at the same time.

With (i) and (ii) we get:

Distinguishable particles. Their 'configuration manifold', denoted by $D_n(P)$, is $(P \times \cdots \times P) \setminus \Delta$, with the diagonal Δ as the set of points $(x_1, \ldots, x_n) \in P \times \cdots \times P$, where $x_i = x_j$ for some $i \neq j$.

The removal of Δ from $P \times \cdots \times P$ has consequences for the topological classification of line bundles and allows the following construction for:

Identical particles. Consider the right group action r of the permutation group S_n:

$$r : D_n(P) \times S_n \longrightarrow D_n(P),$$
$$r_\sigma(x_1, \ldots, x_n) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n$$

(i.e. $r_{\sigma'} r_\sigma = r_{\sigma\sigma'}$, corresponding to standard definition of multiplication in S_n). Since r is a free and discontinuous action, the quotient

$$I_n(P) := D_n(P)/S_n$$

is a smooth manifold with smooth projection $D_n(P) \longrightarrow I_n(P)$, and it yields the 'configuration manifold' for n identical particles on P. $D_n(P)$ is an S_n-bundle over $I_n(P)$ (a p.f.b. with structure group S_n; see e.g. [12]). $\tilde{I}_n(P)$ denotes the universal covering of $I_n(P)$, which of course is also the universal covering of $D_n(P)$ ($\dim M \geq 2$).

For our quantization procedure outlined in Sections 1 and 2, the interesting geometric objects are *flat Hermitian line bundles* $l(M)$ for $M = D_n(P), I_n(P)$ and their *fundamental groups*. These objects have the following properties:

The fundamental groups of $I_n(P)$ and $D_n(P)$, denoted by $B_{n,P} := \pi_1(I_n(P))$ and $C_{n,P} := \pi_1(D_n(P))$, respectively, are *generalized* braid and coloured braid groups (for $P = \mathbb{R}^2$ we get the usual *braid groups* $B_{n,\mathbb{R}^2} = B_n$ and *coloured braid groups* $C_{n,\mathbb{R}^2} = C_n$).

For the h.l.b. we have

(i) the sequence of bundle projections between p.f.b.

$$\tilde{I}_n(P) \longrightarrow D_n(P) \longrightarrow I_n(P), \tag{11}$$

with $\tilde{I}_n(P)$ as a $C_{n,P}$-bundle over $D_n(P)$ and at the same time as a $B_{n,P}$-bundle over $I_n(P)$, while $D_n(P)$ is an S_n-bundle over $I_n(P)$, and

(ii) the exact sequence of groups

$$\{\mathbf{1}\} \to C_{n,P} \to B_{n,P} \to S_n \to \{\mathbf{1}\},$$

i.e. $C_{n,P}$ is an invariant subgroup of $B_{n,P}$, and $S_n = B_{n,P}/C_{n,P}$.

Concerning *flat connections* ∇ on h.l.b., we mentioned already that the equivalence classes of all pairs $(l(M), \nabla)$ for given M are classified by $\pi_1^*(M)$. For their explicit construction take the simply connected covering $\tilde{M}$ of M, which is a $\pi_1(M)$-bundle over M. Then for each homomorphism $\alpha \in \pi_1^*(M)$, consider α as the (left) representation $\pi_1(M) \times \mathbb{C} \to \mathbb{C}$, $(a, z) \mapsto \alpha(a)z$, of $\pi_1(M)$ in $\mathbb{C}$. Define the line bundle $l_\alpha(M)$ over M, α-associated to the $\pi_1(M)$-bundle $\tilde{M}$ over M:

$$l_\alpha(M) := (\tilde{M} \times \mathbb{C})/(\pi_1(M) \times \alpha^{-1}(\pi_1(M))) \tag{12}$$

(short notation: $l_\alpha(M) := (\tilde{M} \times \mathbb{C})/\pi_1(M)$).

If $\tilde{\nabla}$ is the standard flat connection in the line bundle $\tilde{M} \times \mathbb{C}$, this factorization yields also a flat connection ∇_α on $l_\alpha(M)$. By construction, pairs $(l_\alpha(M), \nabla_\alpha)$ are nonequivalent for different α and one gets all flat line bundles over M in this way.

Now, taking $M = I_n(P)$, the factorization with respect to each $\alpha \in \pi_1^*(I_n(P)) = B_{n,P}^*$:

$$\tilde{I}_n(P) \times \mathbb{C} \longrightarrow l_\alpha(I_n(P)) = (\tilde{I}_n(P) \times \mathbb{C})/B_{n,P},$$

splits, corresponding to (11), in a natural way into two steps and gives the line bundles over $D_n(P)$ and $I_n(P)$ (including the flat connections):

(i) *Line bundles over $D_n(P)$*. Consider $\tilde{I}_n(P)$ as the $C_{n,P}$-bundle over $D_n(P)$. Then $\alpha \in \pi_1^*(I_n(P))$ induces the character $\alpha \mid \pi_1(D_n(P)) =: \alpha_D \in \pi_1^*(D_n(P))$. Factorization of $\tilde{I}_n(P) \times \mathbb{C}$ with respect to α_D yields a flat h.l.b. $l_{\alpha_D}(D_n(P)) = (\tilde{I}_n(P) \times \mathbb{C})/C_{n,P}$ over the configuration manifold $D_n(P)$.

(ii) *Line bundles over $I_n(P)$*. Since $C_{n,P}$ is an invariant subgroup of $B_{n,P}$, we have an α-induced right action of $S_n = B_{n,P}/C_{n,P}$ on $l_{\alpha_D}(D_n(P))$ and get $l_{\alpha_D}(D_n(P))$ as an S_n-bundle over $l_\alpha(I_n(P))$ and

$$l_\alpha(I_n(P)) = l_{\alpha_D}(D_n(P))/S_n \tag{13}$$

as a h.l.b. over the configuration manifold $I_n(P)$ for n identical particles.

The topological interpretation of the last fact is the following: For each $\alpha \in \pi_1^*(I_n(P))$ the symmetry properties of 'wave functions' on $D_n(P)$ (sections in $l_{\alpha_D}(D_n(P))$) are given by the right action of S_n on $l_{\alpha_D}(D_n(P))$; hence they are encoded in the construction of $l_\alpha(I_n(P))$. Correspondingly, the topological effect of an exchange of two identical particles, localized around two distinct points in P can be interpreted as the result of a parallel transport (with respect to the α-induced connection in $l_\alpha(I_n(P))$) along a noncontractible closed curve c in $I_n(P)$ (with nonclosed covering $\bar{c}$ in $D_n(P)$). This yields a phase shift for the corresponding 'wave function' on $I_n(P)$ (section in $l_\alpha(I_n(P))$). This particle exchange one may call 'pure', if there is no extra topological effect stemming from the nontrivial topology of $D_n(P)$, i.e. the exchange is pure if the lift $\bar{c}$ of c has no noncontractible closed parts.

Locally, over an open contractible $U \subset I_n(P)$, the two-step factorization of $\tilde{I}_n(P) \times \mathbb{C}$ is given by

$$(U \times B_{n,P}) \times \mathbb{C} \xrightarrow{/C_{n,P}} (U \times S_n) \times \mathbb{C} \xrightarrow{/S_n} U \times \mathbb{C}.$$

Summarizing these facts, we get a commutative diagram for line bundles over configuration spaces (and their universal coverings) of n particles on a physical space P, with vertical projections to the base spaces $\mathcal{B}$ and horizontal bundle factorizations, where not only the base spaces are p.f.b. (over $I_n(P)$), but also the corresponding line bundles, with the same structure groups (over $l_\alpha(I_n(P))$). We add to the diagram two lines: the structure group of $\mathcal{B}$ as a p.f.b. and the fundamental group of $\mathcal{B}$:

line bundle	$\tilde{I}_n(P) \times \mathbb{C}$	$\xrightarrow{/C_{n,P}}$	$l_{\alpha_D}(D_n(P))$	$\xrightarrow{/S_n}$	$l_\alpha(I_n(P))$
	$\downarrow$		$\downarrow$		$\downarrow$
base space $\mathcal{B}$	$\tilde{I}_n(P)$	$\xrightarrow{/C_{n,P}}$	$D_n(P)$	$\xrightarrow{/S_n}$	$I_n(P)$
structure group of p.f.b. $\mathcal{B}$ *over* $I_n(P)$	$B_{n,P}$	$\xrightarrow{/C_{n,P}}$	S_n	$\xrightarrow{/S_n}$	$\{1\}$
fund. group of $\mathcal{B}$	$\{1\}$		$C_{n,P}$		$B_{n,P}$

4. Application to Borel quantization for n particles on a 2-manifold

The Borel quantization assumes that the (topological) structure of the 'classical' configuration manifold M survives the quantization map $\mathcal{Q}$. Hence we can use the above geometrical results to classify nonequivalent quantizations on $\mathbb{R}^2$ and on compact orientable 2-manifolds as physical spaces P. In Section 4.1 the characters of $\pi_1(M)$ are calculated. Here the commutativity of $U(1)$ simplifies the homomorphic images of the algebraic relations characterizing $\pi_1(M)$ considerably. To get the kinematical operators $\mathbb{Q}(f)$ and $\mathbb{P}(X)$ we give the pairs $(l(M), \nabla)$ of flat line bundles and the corresponding Hilbert spaces in Section 4.2 for $P = \mathbb{R}^2$, and (see Eq. (10)) the (linear) Schrödinger equation along the lines explained before in Section 4.3.

4.1. U(1)-representations of the fundamental groups

As mentioned in Section 3, the fundamental groups of our configuration spaces are generalizations of usual braid groups B_n and coloured braid groups C_n. They are given through natural geometric constructions [3,5,13,22], i.e. the topological holonomy effects in M arising from a continuous exchange of the n-particle configurations are translated into algebraic relations in the free group F_l of l generators, l depending on n. Factorization of F_l by these relations then yields the generalized (coloured) braid groups. In particular, for the case of $P = \mathbb{R}^2$, this gives B_n and C_n [3,13].

We list the fundamental groups together with the defining relations of their generators and their inequivalent $U(1)$-representations for $\mathbb{R}^2$ and for compact orientable two-dimensional physical spaces P.

4.1.1. n Distinguishable particles in $\mathbb{R}^2$

(i) *Fundamental groups.* $\pi_1(D_n(\mathbb{R}^2))$ coincides with the nth coloured braid group C_n [3]. C_n is defined via the free group F_l of $l = n(n-1)/2$ generators A_{ij}, $i, j \in \{1, \ldots, n\}$, $i < j$, and their inverses A_{ij}^{-1}, together with the complete set of defining relations [3]

$$
\begin{aligned}
A_{rs}A_{ik}A_{rs}^{-1} &= A_{ik}, \quad 1 \le i < k < r < s \le n, \text{ or } 1 \le i < r < s < k \le n,\\
A_{rs}A_{ir}A_{rs}^{-1} &= A_{is}^{-1}A_{ir}A_{is}, \quad 1 \le i < r < s \le n,\\
A_{rs}^{-1}A_{ir}A_{rs} &= A_{is}A_{ir}A_{is}^{-1}, \quad 1 \le i < s < r \le n,\\
A_{rs}A_{is}A_{rs}^{-1} &= A_{is}^{-1}A_{ir}^{-1}A_{is}A_{ir}A_{is}, \quad 1 \le i < r < s \le n,\\
A_{rs}^{-1}A_{is}A_{rs} &= A_{is}A_{ir}A_{is}A_{ir}^{-1}A_{is}^{-1}, \quad 1 \le i < s < r \le n,\\
A_{rs}A_{ik}A_{rs}^{-1} &= A_{is}^{-1}A_{ir}^{-1}A_{is}A_{ir}A_{ik}A_{ir}^{-1}A_{is}^{-1}A_{ir}A_{is}, \quad 1 \le i < r < k < s \le n,\\
A_{rs}^{-1}A_{ik}A_{rs} &= A_{is}A_{ir}A_{is}^{-1}A_{ir}^{-1}A_{ik}A_{ir}A_{is}A_{ir}^{-1}A_{is}^{-1}, \quad 1 \le i < s < k < r \le n.
\end{aligned}
$$

(ii) *Characters.* The inequivalent $U(1)$-representations of $\pi_1(D_n(\mathbb{R}^2))$ are classified by the numbers ζ in $\prod_{\substack{i,j=1\\ i<j}}^{n}[0,2\pi)_{ij} \equiv [0,2\pi)^{n(n-1)/2}$.

For a proof of (ii) observe that, according to the universal property of free groups, each homomorphism $\alpha_D : C_n \to U(1)$ is a realization of A_{ij} in $U(1)$, such that the defining relations are fulfilled; commutativity of $U(1)$ yields only trivial relations in $U(1)$ in the present case. Hence each element $\alpha_D(A_{ij}) = \exp \mathrm{i}\zeta \in U(1)$ can be chosen arbitrarily. Different choices give inequivalent representations. Hence $\pi_1^*(D_n(\mathbb{R}^2)) = \prod^{n(n-1)/2} U(1)$.

(iii) *Line bundles with flat connections.* The only line bundles over $D_n(\mathbb{R}^2)$ with flat connections are trivializable, i.e. bundle isomorphic to $D_n(\mathbb{R}^2) \times \mathbb{C}$.

To show this, observe that $\pi_1^*(D_n(\mathbb{R}^2))$ characterizes the equivalence classes of pairs $(l_\alpha(D_n(\mathbb{R}^2)), \nabla_\alpha)$ with flat ∇_α, $\alpha \in \pi_1^*(D_n(\mathbb{R}^2))$. On the other hand, in Section 4.2 we explicitly list all these equivalence classes for trivial (and hence for trivializable) $l_\alpha(D_n(\mathbb{R}^2))$. These pairs already exhaust $\pi_1^*(D_n(\mathbb{R}^2))$, i.e. there are no others.

4.1.2. *n Identical particles in* $\mathbb{R}^2$

(i) *Fundamental groups.* $\pi_1(I_n(\mathbb{R}^2))$ coincides with the nth braid group B_n [3,13]. B_n is generated by $b_1, \ldots, b_{n-1}$ and the complete set of defining relations [3,13]

$$
\begin{aligned}
b_i b_j &= b_j b_i, \quad |i-j| \ge 2,\\
b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \quad i = 1, \ldots, n-2.
\end{aligned}
$$

(ii) *Characters.* The inequivalent $U(1)$-representations of $\pi_1(I_n(\mathbb{R}^2))$ are classified by the numbers in $[0, 2\pi)$.

To prove (ii), observe, that the nontrivial realizations of the defining relations in $U(1)$ are only $b_i = b_{i+1}$, $i = 1, \ldots, n-2$, hence $\pi_1^*(D_n(\mathbb{R}^2)) = U(1)$.

(iii) *Line bundles with flat connections.* All line bundles over $I_n(\mathbb{R}^2)$ with flat connections are trivializable. (Same arguments as in Section 4.1.1.)

In Sections 4.1.3 and 4.1.4 we discuss compact orientable 2-manifolds (classified up to homeomorphisms by their genus g). We only list the generators of the fundamental groups and the *nontrivial* realizations of the defining relations in $U(1)$. The computation of the character groups is straightforward and similar as in Sections 4.1.1 and 4.1.2 (see [16]).

4.1.3. *n Distinguishable particles on compact orientable 2-manifolds*

(i) *Fundamental groups.* We call $\pi_1(D_n(P))$ the *'generalized coloured braid group'* $C_{n,P}$ (see Section 3).

Case $g \geq 1$ [5]. $C_{n,P}$ is generated by $\rho_{i1}, \rho_{i2}, \ldots, \rho_{ig}, \tau_{i1}, \ldots, \tau_{ig}$, $i = 1, \ldots, n$. All defining relations for $C_{n,P}$ become trivial in $U(1)$.

Case $g = 0$ [26]. C_{n,S^2} is generated by A_{ij}, $i, j = 1, \ldots, n$, $i < j$. The only remaining nontrivial relation in $U(1)$ is $1 = A_{1,n}A_{1,n-1}, \ldots, A_{1,3}A_{1,2}$.

(ii) *Characters.* The inequivalent $U(1)$-representations of $\pi_1(D_n(P))$ are classified by the numbers in $\prod_{i=1}^{2ng}[0, 2\pi)_i$ for $g \geq 1$, and in $[0, 2\pi)_{1,3} \times [0, 2\pi)_{1,4} \times \cdots \times [0, 2\pi)_{n-1,n}$ for $g = 0$, i.e. for $P = S^2$. There are $(n(n-1)/2) - 1$ independent intervals in the last case.

4.1.4. *n Identical particles on compact orientable 2-manifolds*

(i) *Fundamental groups (see also* [30]). We call $\pi_1(I_n(P))$ the *'generalized braid group'* $B_{n,P}$ (see Section 3).

Case $g \geq 1$ [19]. $B_{n,P}$ is generated by $b_1, \ldots, b_{n-1}, \tau_1, \ldots \tau_g, \rho_1, \ldots, \rho_g$. The nontrivial relations in $U(1)$ are $b_i = b_{i+1}$, $i = 1, \ldots, n-2$, and $b_1^2 = 1$.

Case $g = 0$ [13]. $B_{n,S^2} = \pi_1(I_n(S^2))$ is generated by $b_1, \ldots, b_{n-1}$. The nontrivial relations in $U(1)$ are $b_i = b_{i+1}$, $i = 1, \ldots, n-2$, and $b_1^{2(n-1)} = 1$.

(ii) *Characters.* The inequivalent U(1)-representations of $B_{n,P}$ are characterized by the numbers in $\{-1, +1\} \times \prod_{i=1}^{2g}[0, 2\pi)_i$ for $g \geq 1$ and in $\mathbb{N}\pi/(n-1) \cap [0, 2\pi)$ for $g = 0$ $(P = S^2)$.

4.1.5. *Results*

We collect the results for the $U(1)$-representations of the fundamental groups for distinguishable and identical particles on P, i.e. for $D_n(P)$ and $I_n(P)$, respectively (for the case of $\mathbb{R}^m$, $m \geq 3$, see Appendix A):

	$P = \mathbb{R}^2$	P compact, $\dim P = 2$, $g = 0$	P compact, $\dim P = 2$, $g \geq 1$	$P = \mathbb{R}^m, m \geq 3$
$D_n(P)$	$[0, 2\pi)^{n(n-1)/2}$	$[0, 2\pi)^{(n(n-1)/2)-1}$	$[0, 2\pi)^{2ng}$	1
$I_n(P)$	$[0, 2\pi)$	$\mathbb{N}\pi/(n-1) \cap [0, 2\pi)$	$\{-1, 1\} \times [0, 2\pi)^{2g}$	$\{-1, 1\}$

4.2. *Representations of the kinematical algebra $S(M)$ for $\mathbb{R}^2$ as physical space*

With the results in Section 2.1 we construct representations of $S(M)$ up to unitary equivalence for $P = \mathbb{R}^2$ in the Hilbert space $L^2(l(M), \mathrm{d}\mu)$, $M = D_n(\mathbb{R}^2), I_n(\mathbb{R}^2)$. By construction $\mathbb{Q}(f)$ acts as a multiplication operator f on some dense set $\vartheta \subset L^2(l(M), \mathrm{d}\mu)$. For $\mathbb{P}(X)$, $X \in \mathrm{Vec_c}(M)$, which depends on the flat connection ∇ on $l(M)$, we give the result for $n = 2$ in Section 4.2.1 and for $n > 2$ in Section 4.2.2.

4.2.1. $\mathbb{P}(X)$ *for 2-particle systems*

$M = D_2(\mathbb{R}^2)$. Use in $P \times P$, $P = \mathbb{R}^2$, coordinates x_α^i; with α: particle index, $\alpha = 1, 2$, and i: coordinate index, $i = 1, 2$.

We mentioned in Sections 4.1.1 and 4.1.2 that the existence of flat connections ∇ on $l(M)$ implies $l(D_2(\mathbb{R}^2))$ to be trivializable. Following Section 2.1, each ∇ can be described globally (through a trivializing section in $l(M)$) by the corresponding connection 1-form ω on M. Two unitary (gauge) equivalent ω' and ω'' are related through a logarithmic exact 1-form via ($g : M \to U(1)$)

$$\frac{\mathrm{i}}{\hbar}(\omega' - \omega'') = g^{-1}\,\mathrm{d}g. \tag{14}$$

Consider now the flat connections ${}^D\nabla^\zeta$ in $l(D^2(\mathbb{R}^2))$, which are parametrized through $\zeta \in [0, 2\pi)$, and are given via the closed connection 1-forms (with $x_i = (x_i^1, x_i^2)$):

$$\begin{aligned} &{}^D\omega^\zeta(x_1^1, \ldots, x_2^2) := \sum_{i,\alpha=1,2} {}^D\omega_\alpha^{i,\zeta}\,\mathrm{d}x_\alpha^i, \\ &{}^D\omega_1^{1,\zeta} = -{}^D\omega_2^{1,\zeta} = -\frac{\zeta}{2\pi}\frac{(x_1^2 - x_2^2)}{|x_1 - x_2|^2}, \quad {}^D\omega_1^{2,\zeta} = -{}^D\omega_2^{2,\zeta} = \frac{\zeta}{2\pi}\frac{(x_1^1 - x_2^1)}{|x_1 - x_2|^2}. \end{aligned} \tag{15}$$

To find the inequivalent ${}^D\omega^\zeta$ we select a convenient $\hat{X} \in \mathrm{Vec}(M)$ and show that Eq. (14), evaluated on $\hat{X}$, enforces $\zeta' - \zeta'' = 2\pi m, m \in \mathbb{Z}$. We introduce coordinates $x_S^i = \frac{1}{2}(x_1^i + x_2^i)$, $x_R^i = \frac{1}{2}(x_1^i - x_2^i)$, $i = 1, 2$, on $D^2(\mathbb{R}^2) = \mathbb{R}^2_{x_S} \times \dot{\mathbb{R}}^2_{x_R}$:

$${}^D\omega^\zeta = \frac{1}{2\pi}\frac{\zeta}{|x_R|^2}(-x_R^2\,\mathrm{d}x_R^1 + x_R^1\,\mathrm{d}x_R^2),$$

and we select $\hat{X}$ as the infinitesimal rotation in $\mathbb{R}^2_{x_R}$, i.e. $\hat{X} = x_R^1(\partial/\partial x_R^2) - x_R^2(\partial/\partial x_R^1) = (\partial/\partial\phi)$ (ϕ as the polar angle in $\mathbb{R}^2_{x_R}$),

$${}^D\omega^\zeta(X) = \frac{1}{2\pi}\zeta. \tag{16}$$

The restriction to $\{0\} \times S^1 \subset \mathbb{R}^2_{x_S} \times \dot{\mathbb{R}}^2_{x_R}$ now yields for the difference of two connection forms (see Eq. (14)),

$$g^{-1}\,\mathrm{d}g(X) = \mathrm{i}\frac{\partial h}{\partial\phi}, \tag{17}$$

where $g = \exp \mathrm{i}h$, $h : S^1 \to \mathbb{R}$, with $h(2\pi) = h(0) + 2\pi m$, $m \in \mathbb{Z}$.

Eq. (16) implies $h(\phi) - h(0) = (1/2\pi)(\zeta' - \zeta'')\phi$ and $\zeta' - \zeta'' = 2\pi m$, $m \in \mathbb{Z}$. Hence the equivalence classes of all ${}^D\omega^\zeta$ are characterized by the elements $\zeta \in [0, 2\pi)$.

It remains to note that Eq. (15) gives, up to equivalence, *all* different flat connections on $l(M)$. This is because the equivalence classes $(l(M), \nabla)$ in Section 3 with classification through $\pi_1^*(M) = U(1)$ (Sections 4.1.1 and 4.1.2) are formally and technically in correspondence with the above calculation.

$M = I_2(\mathbb{R}^2)$. The 1-form ${}^D\omega^\zeta$ is invariant under S_2; each ${}^D\omega^\zeta$ is the pull back of a unique 1-form ${}^I\omega^\zeta$ on $I_2(\mathbb{R}^2)$. Obviously, ${}^I\omega^{\zeta'}$ and ${}^I\omega^{\zeta''}$ are gauge equivalent iff the corresponding ${}^D\omega^{\zeta'}$ and ${}^D\omega^{\zeta''}$ are. Hence we have the same result for ${}^I\omega^\zeta$ as for the ${}^D\omega^\zeta$.

$\mathbb{P}(X)$. To calculate $\mathbb{P}(X)$, insert ω^ζ in Eqs. (2), (3) and get (the line bundles are trivial), for distinguishable as well as for identical particles, on a dense set in $L^2(l(M),\ \mathrm{d}x_1^1, \dots, \mathrm{d}x_2^2)$,

$$\mathbb{P}^{\zeta,c}(X) = \frac{\hbar}{\mathrm{i}}\left(X + \frac{\mathrm{i}}{\hbar}\omega^\zeta(X) + \frac{1}{2}\operatorname{div} X\right) + c\operatorname{div} X, \tag{18}$$

with $X \in \mathrm{Vec}_c(M)$. For different $\zeta \in [0, 2\pi)$ or different $c \in \hbar\mathbb{R}$ the $\mathbb{P}^{\zeta,c}(X)$, together with the $\mathbb{Q}(f)$, are unitarily inequivalent representations of $S(M)$.

4.2.2. $\mathbb{P}(X)$ for n-particle systems, $n > 2$

$M = D_n(\mathbb{R}^2)$. Use in $\mathbb{R}_1^2 \times \cdots \times \mathbb{R}_n^2$ coordinates x_α^i, $\alpha = 1, \dots, n$, $i = 1, 2$, and consider the following connection 1-forms on $D_n(\mathbb{R}^2)$, parametrized through $\zeta := \{\zeta_{\alpha,\beta} \in [0, 2\pi)/\alpha < \beta; \alpha, \beta = 1, \dots, n\}$ $(\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1)$:

$$\begin{aligned} {}^D\omega^\zeta(x_1^1, \dots, x_n^2) &:= \sum_{\substack{i=1,2\\ \alpha=1,\dots,n}} {}^D\omega_\alpha^{\zeta,i}\,\mathrm{d}x_\alpha^i, \\ {}^D\omega_\alpha^{\zeta,i} &= \sum_{\substack{\beta=1,\dots,n:\beta>\alpha\\ l=1,2}} \frac{\zeta_{\alpha,\beta}}{2\pi}\frac{\epsilon_{il}(x_\alpha^l - x_\beta^l)}{|x_\alpha - x_\beta|^2}. \end{aligned} \tag{19}$$

The ${}^D\omega^\zeta$ are invariant under S_n and furthermore closed for all $\zeta_{\alpha,\beta} \in \mathbb{R}$; the corresponding line bundles are trivial. Unitary equivalent ${}^D\omega^\zeta$ are related through a logarithmic exact 1-form (see Eq. (14)). It remains to show that the forms in Eq. (19) for different $\zeta = \{\zeta_{\alpha,\beta}\}$ are pairwise nonequivalent.

We use the results for $n = 2$ and the fact that two closed nonequivalent 1-forms on an open region $B \in M$ are also nonequivalent on M.

Take one of the nonvanishing $\zeta_{\alpha\beta}$, say ζ_{12}. We choose $B = ((U_1 \times U_2) \setminus D_{12}) \times U_3 \times \cdots \times U_n \subset D_n(\mathbb{R}^2)$ with: for $\alpha = 1, 2$ the U_α are open contractible neighbourhoods of $0 \in \mathbb{R}_\alpha^2$, D_{12} as the diagonal in $\mathbb{R}_1^2 \times \mathbb{R}_2^2$; for $\alpha = 3, \dots, n$ the U_α are open contractible disjoint sets in $\mathbb{R}_\alpha^2$, with $U_\alpha \cap U_1 = \emptyset$, $U_\alpha \cap U_2 = \emptyset$. Restrict now ${}^D\omega^\zeta$ to B. The restriction $\tilde{\omega}^\zeta$ then can be written as a sum of a term $\omega^{\zeta_{12}}$ depending on x_1, x_2 only and a rest term $\omega_{\mathrm{rest}}^\zeta$, i.e. $\tilde{\omega}^\zeta = \omega^{\zeta_{12}} + \omega_{\mathrm{rest}}^\zeta$. Since $\omega_{\mathrm{rest}}^\zeta$ is nonsingular not only on B, but also on the contractible set $U_1 \times \cdots \times U_n$, it is exact, $\omega_{\mathrm{rest}}^\zeta = \hbar\,\mathrm{d}h$ (h real), and hence logarithmic exact on B. Thus, it has no influence on the classification of $\tilde{\omega}^\zeta$. The other term $\omega^{\zeta_{12}}$ coincides with the connection form given in Eq. (15) for two identical particles ($\zeta_{12} = \zeta$). This implies on B, and hence on $D_n(\mathbb{R}^2)$, that $\tilde{\omega}^{\zeta'} - \tilde{\omega}^{\zeta''}$ is logarithmic exact iff $\zeta_{12}' - \zeta_{12}'' = 2\pi m$, $m \in \mathbb{Z}$.

$M = I_n(\mathbb{R}^2)$. For ${}^I\omega^\zeta$ we use the arguments for the case $n = 2$ in Section 4.2.1. The S_n symmetry requires $\zeta_{\alpha\beta} = \zeta$, $\alpha < \beta$ in ${}^D\omega^\zeta$. One shows that Eq. (19) (with analogously defined ${}^I\omega_\alpha^{\zeta,i}$, but $\zeta \in [0, 2\pi)$) gives all different flat connections on $l(M)$ via the projection $D_n(\mathbb{R}^2) \to I_n(\mathbb{R}^2)$.

$\mathbb{P}(X)$. The $\mathbb{P}(X)$ are, with ${}^D\omega^\zeta$ and ${}^I\omega^\zeta$, resp., similar to Eq. (18) in the case of two particles. For $M = D_n(\mathbb{R}^2)$ we get on a dense set in $L^2(l(M),\ \mathrm{d}x_1^1, \dots, \mathrm{d}x_n^2)$:

$$\mathbb{P}^{\zeta,c}(X) = \frac{\hbar}{\mathrm{i}}\left(X + \frac{\mathrm{i}}{\hbar}\,{}^{D}\omega^{\zeta}(X) + \frac{1}{2}\operatorname{div} X\right) + c \operatorname{div} X, \tag{20}$$

with $X \in \mathrm{Vec}_c(\mathbb{R})$.

For different $\zeta = \{\zeta_{\alpha\beta} \in [0, 2\pi)/\alpha < \beta; \alpha, \beta = 1, \ldots, n\}$ or different $c \in \hbar\mathbb{R}$ the $\mathbb{P}(X)$, together with the $\mathbb{Q}(f)$, are unitarily inequivalent representations of $S(M)$.

If one replaces ${}^{D}\omega^{\zeta}$ by ${}^{I}\omega^{\zeta}$ and $D_n(\mathbb{R}^2)$ by $I_n(\mathbb{R}^2)$, with $\zeta \in [0, 2\pi)$, one gets the corresponding result for identical particles.

4.3. Evolution equations for $\mathbb{R}^2$ as physical space

The last step in the Borel quantization is the description of an evolution (Schrödinger) equation for our system. We discuss n particles on $P = \mathbb{R}^2$ with masses $m_\alpha, \alpha = 1, \ldots, n$, where $m_\alpha = m$ in the case of identical particles. We use the above results. $D_n(\mathbb{R}^2)$ and $I_n(\mathbb{R}^2)$ are furnished with a Riemannian metric g induced from $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 : g_{(i,\alpha)(j,\beta)} = m_\alpha \delta_{ij} \delta_{\alpha\beta}$ and $g_{(i\alpha)(j\beta)} = m \delta_{ij} \delta_{\alpha\beta}$, respectively. Insert this g and the connection forms ${}^{I}\omega^{\zeta}$ and ${}^{D}\omega^{\zeta}$ in Eqs. (8), (9) and get for $c = 0$ and $R = 0$ a linear Schrödinger equation with potential V for

(i) n distinguishable particles on $\mathbb{R}^2$:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi = \sum_{\substack{i=1,2 \\ \alpha=1,\ldots,n}} \frac{-\hbar^2}{2m_\alpha}\left(\frac{\partial}{\partial x^i_\alpha} + \frac{\mathrm{i}}{\hbar}\,{}^{D}\omega^{i,\zeta}_{\alpha}\right)^2 \psi + V\psi, \tag{21}$$

where $\psi \in L^2(D_n(\mathbb{R}^2), \mathrm{d}x^1_1, \ldots, \mathrm{d}x^2_n)$, $\zeta = \{\zeta_{\alpha,\beta}\}$, and for

(ii) n identical particles on $\mathbb{R}^2$:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi = \sum_{\substack{i=1,2 \\ \alpha=1,\ldots,n}} \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial x^i_\alpha} + \frac{\mathrm{i}}{\hbar}\,{}^{I}\omega^{i,\zeta}_{\alpha}\right)^2 \psi + V\psi, \tag{22}$$

with $\psi \in L^2(D_n(\mathbb{R}^2), \mathrm{d}x^1_1, \ldots, \mathrm{d}x^2_n)$, $\zeta \in [0, 2\pi)$, and with S_n-invariant potential V.

The $\omega^{\zeta,i}_\alpha$ are (gauge) potentials. The choice of ζ characterizes the quantum mechanics of the n distinguishable or identical particles on $\mathbb{R}^2$ up to unitary equivalence. Some properties of these quantum systems are known. In the case of identical particles the constituents are 'anyons' (see Section 1). Eq. (22) is the n-anyon Schrödinger equation, which was derived with a path integral quantization or viewed as a Chern–Simons dynamics (see e.g. [22]). This equation was extended to many particle systems by a mean field approximation [20]. Here we presented a strict geometric derivation. Solutions of the n-anyon Schrödinger equation for the harmonic oscillator or the electromagnetic vector potential are known (see e.g. [6]).

The nonlinear term proportional to c and the R-term in Eq. (8) are independent of the topology of M. Hence a nonlinear version of an n-anyon equation contains a nonlinear term $c(\Delta_g \rho/\rho)$ and R depending on ψ, $\bar{\psi}$ and their derivatives.

5. Concluding remarks

Our approach shows the possible physical relevance and especially the geometrical richness of n-particle quantum mechanics on smooth *two*-dimensional spaces P. We presented a detailed description for $P = \mathbb{R}^2$ and discussed compact orientable P. Other two-dimensional manifolds can be treated along the same lines, in particular the Klein-bottle, the torus and the projective plane (these manifolds can be viewed as quotient spaces of $\mathbb{R}^2$), furthermore the N pointed $\mathbb{R}^2$. However, in general the corresponding line bundles are not trivial, and a general method to calculate the connections explicitly and globally in a straightforward way is not at hand. For compact spaces P with dimension $m > 2$ we quote the result [17]

$$\pi_1(D_n(P)) = (\pi_1(P))^n, \qquad \pi_1(I_n(P)) = S_n \otimes_s (\pi_1(P))^n,$$

which shows that, in contrast to $m = 2$, topological pecularities are connected with $\pi_1(P)$ only.

Our study is restricted to line bundles, which carry a one-dimensional representation of $\pi_1(M)$. In principle also vector bundles ($\mathbb{C}^k$-bundles) can be used. Here k-dimensional representations of the fundamental groups appear, e.g. for $P = \mathbb{R}^2$ k-dimensional representations of C_n and B_n; the corresponding constituents of the n-particle system were called plectons [25].

Generally, in a quantization based on $\mathbb{C}^k$-bundles topological effects of 'type I and type II' are known [23,27], which are due to $\pi_1(M)$ and the Čech-cohomology groups $H^1_C(M, \underline{U(1)})$ with smooth $U(1)$-valued functions as coefficients, respectively. Here type I means effects, which are due to the classification of $\mathbb{C}^k$-bundles *with* given flat connection, and type II concerns effects depending on the topology of the $\mathbb{C}^k$-bundle *without* specification of a flat connection. Only type I effects are discussed in the present paper. For *line* bundles with $P = \mathbb{R}^2$ in the case of two distinguishable or two identical particles there are no type II effects because $H^1_C(M, \underline{U(1)}) = H^2(M, \mathbb{Z}) = 0$, but nonequivalent flat line bundles exist, i.e. effects of type I appear.

Acknowledgements

We are indepted to G.A. Goldin, W. Lücke and H.-J. Mann for helpful discussions. WG wants to thank the state of Lower Saxony for financial support through the Graduiertenförderung.

Appendix A. Applications to $P = \mathbb{R}^m, m \geq 3$

Borel quantization can be applied to n particles on any smooth P. We give the (known) results for $P = \mathbb{R}^m$, $m \geq 3$, for a comparison with our study in the case of $P = \mathbb{R}^2$.

(i) *Standard statistics.* For $P = \mathbb{R}^m$, $m \geq 3$, $D_n(P)$ is simply connected, hence $C_{n,P} = \{\mathbf{1}\}$, $B_{n,P} = S_n$, and we get a unique line bundle $l_\alpha(D_n(P)) = D_n(P) \times \mathbb{C}$, $\alpha \in \{\mathbf{1}\}^*$, with standard flat connection for the case of distinguishable particles (cf. the diagram in Section 3).

For identical particles $B_n^*(P) = S_n^* = \mathbb{Z}_2$ holds, and $l_\alpha(I_n(P))$ gives two cases:
$l_\alpha(I_n(P)) = I_n(P) \times \mathbb{C}$ for the trivial $\alpha \in \mathbb{Z}_2$, and
$l_\alpha(I_n(P)) \neq I_n(P) \times \mathbb{C}$ for the nontrivial $\alpha \in \mathbb{Z}_2$.
The first corresponds to *Boson* and the second to *Fermion statistics*.

In comparison to the case of $P = \mathbb{R}^2$ (see the interpretation following Eq. (13)), the symmetry properties of 'wave functions' on $D_n(\mathbb{R}^m)$, $m \geq 3$, which here are sections ψ of $l_\alpha(I_n(\mathbb{R}^m))$, are given by the *representation* of S_n in $U(1)$, since $l_{\alpha_D}(D_n(\mathbb{R}^m)) = D_n(\mathbb{R}^m) \times \mathbb{C}$ (see Section 3). Hence the exchange of two identical particles, interpreted via parallel transport along a closed curve in $I_n(P)$, yields no phase shift (Boson case) or a phase shift -1 (Fermion case) for ψ.

(ii) *Parastatistics.* Vector bundles over $I_n(P)$ corresponding to higher dimensional representations of S_n are interpreted as parastatistics and appear in this approach as follows. Consider the configuration space $D_n(P)$. The right action r of S_n on $D_n(P)$, $r : D_n(P) \times S_n \to D_n(P)$, induces via pull back a natural left action T of S_n on the space of smooth square integrable sections $\psi \in \sec^\infty(D_n(P) \times \mathbb{C})$, i.e. with $s \in S_n$, $(T(s)\psi)(x) := (r^*_{s^{-1}}\psi)(x)$.

Close this space to the n-particle Hilbert space $\mathcal{H}$. In $\mathcal{H}$ the action T is completely reducible, $T = \oplus T_\lambda$, with T_λ irreducible and $\mathcal{H}_\lambda$ as the corresponding subspace of $\mathcal{H}$. The reduction gives $(r_s^*\psi)(x) = (T_\lambda(s^{-1})\psi)(x)$ for $\psi \in \mathcal{H}_\lambda \cap \sec^\infty(D_n(P) \times \mathbb{C})$, i.e. the ψ are T_λ-equivariant functions on $D_n(M)$ with 'fibre' $\mathcal{H}_\lambda = \mathbb{C}^k$.

Collect all representations equivalent to T_λ for fixed λ. Then the set of equivariant ψ generates a $\mathbb{C}^k$-bundle, which is T_λ-associated to the S_n bundle $D_n(P)$ over $I_n(P)$.

Alternatively, to get the same result, one can consider the S_n-bundle $D_n(P)$ over $I_n(P)$ fibrewise, i.e. restrict the ψ's to each single S_n-orbit in $D_n(P)$. Then the action T yields the *regular* $n!$-dimensional complex representation T_{reg} of S_n. Each irreducible part T_λ of T_{reg} now defines a $\mathbb{C}^k$-bundle, T_λ-associated to $D_n(P)$.

For $k = 1$ we recover the two line bundles in (i) (Boson and Fermion case).
For $k > 1$ we get *parastatistics*. Since we are concerned in this paper with *line* bundles only, we postpone this case and the corresponding generalizations to arbitrary P and arbitrary dimensions for later investigations.

Note that in quantum Borel kinematics $\mathbb{C}^k$-bundles also appear in a physically and technically different context. If one assumes that the system has k internal degrees of freedom one has to realize $\mathcal{H}$ via square integrable sections in $\mathbb{C}^k$-bundles over M [24], using similar arguments as in Section 1. In this case the classification of flat bundles is given via the set of conjugacy classes in $\text{Hom}(\pi_1(M), U(k))$.

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